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AN INTEGRAL REPRESENTATION FOR EIGENFUNCTIONS OF LINEAR ORDINAR--ETC(U)
JAN 78 W SYMES DAAG29-75-C-0024
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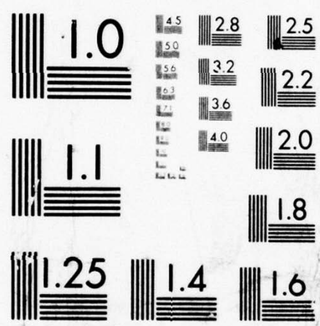
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AN INTEGRAL REPRESENTATION FOR EIGENFUNCTIONS OF LINEAR ORDINARY DIFFERENTIAL OPERATORS.

10 W. Symes

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AN INTEGRAL REPRESENTATION FOR EIGENFUNCTIONS OF LINEAR ORDINARY DIFFERENTIAL OPERATORS

W. Symes

Technical Summary Report #1824
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ABSTRACT

This paper generalizes an integral representation formula for eigenfunctions of Sturm-Liouville operators, known as the Volterra transformation operator in the theory of the inverse scattering problem, to higher-order differential operators. A specific fourth-order initial value problem is considered:

$$L\phi = k^4\phi \,, \quad L = \frac{d^4}{dx^4} + \frac{d}{dx}\left(q\frac{d}{dx}\right) + r$$

$$\phi(o) = 1 \,, \quad \phi'(o) = 0 \,, \quad \phi''(o) = -k^2 \,, \quad \phi'''(o) = 0$$

The solution for complex k is expressed as an inverse-Laplace-Borel transform. Jump formulae are obtained relating the representing kernel directly to the coefficients of L. The result admits obvious generalization to operators of arbitrary order.

AMS (MOS) Subject Classification: 34B25

Key words: Linear initial value problem with complex parameter, Laplace-Borel Transform, Entire functions of exponential type.

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SIGNIFICANCE AND EXPLANATION

The motions of systems governed by linear ordinary differential equations can often be developed in series of normal modes or eigenfunctions; i.e., solutions of initial- or boundary-value problems involving differential equations of the form

$$L\phi = \lambda\phi$$

where L is a linear ordinary differential operator, and λ is a complex number.

In the theory of <u>inverse spectral problems</u> — that is, problems in which properties of a differential operator or system are to be inferred from information about the normal mode expansion — detailed information about the structure of normal modes plays a role. For instance, the density of a vibrating string can be deduced from observation of the motion of a single point on the string, which in turn may be interpreted as information about the normal modes. In the solution of this problem, an integral transform of Volterra type, relating the normal modes of an arbitrary string to trigonometric functions (normal modes of a homogeneous string), plays a central role.

The differential operator which figures in the string problem is of second-order. Inverse problems involving higher-order operators, however, have so far resisted analysis. Among these is the problem of inferring the density of a vibrating beam from the motion of one point on the beam, which again may be interpreted as normal mode information. This inverse problem involves a differential operator of fourth order.

This paper describes an integral representation formula for eigenfunctions of higher order — in particular, fourth order — operators, which has many points in common with the Volterra transform mentioned above in connection with the inverse problem of the vibrating string. The Volterra transform is in fact a special case, and the representing kernel is directly related to the coefficients of the operator involved.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

AN INTEGRAL REPRESENTATION FOR EIGENFUNCTIONS OF LINEAR ORDINARY DIFFERENTIAL OPERATORS

W. Symes

§1. Introduction and Statement of Results

The purpose of this short note is to present a type of integral representation formula for the solutions of linear differential equations containing a parameter of the form

$$L\phi = \lambda\phi \ , \ L = \sum_{j=0}^{n} q_k \left(-i\frac{d}{dx}\right)^k \ .$$

The main result is a representation formula for the solution of the fourth order initial value problem

$$L\left(x,\frac{d}{dx}\right)\phi(x,k) = k^4\phi(x,k) \qquad x \geq 0$$

$$L\left(x,\frac{d}{dx}\right) = \frac{d^4}{dx^4} + \frac{d}{dx}\left(q(x)\frac{d}{dx}\right) + r(x)$$

$$\phi(o,k) = 1 \ , \ \frac{d\phi}{dx}(o,k) = o \ , \ \frac{d^2\phi}{dx^2}(o,k) = -k^2 \ , \ \frac{d^3\phi}{dx^3}(o,k) = 0. \qquad (1)$$

Here q, q' and r are continuous and complex-valued on $[0, \infty]$.

<u>Representation Theorem</u>. Denote by $P(x)$ the diamond with vertices at $\pm x, \pm ix$. Then the solution $\phi(x,k)$, $x \geq 0$, of the initial value problem (1) admits the integral representation

$$\phi(x,k) = \cos kx + \frac{1}{2\pi i}\int_{\partial P(x)} dz\ e^{kz}\ A(x,z) \qquad (2)$$

The kernel $A(x,z)$ is square-integrable on $z\varepsilon\partial P(x)$, and continuous for $x \geq 0$, $z\varepsilon\partial P(x)$, $z \neq \pm ix$; at the vertices $\pm ix$, $A(x,z)$ undergoes a jump given by

$$\lim_{\delta\to o}[A(x, \pm ix-\delta) - A(x, \pm ix + \delta)] = -\frac{\pi}{4}\int_o^x dt q(t) \qquad (3)$$

$A(x,z)$ enjoys the symmetry properties

$$A(x, -z) = -A(x,z) \ , \ A(x, \bar{z}) = -\overline{A(x,z)} \qquad (4)$$

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The proof is presented as follows. Several elementary estimates for the solution of (4) are given in §2. The main step in the proof is furnished by a Theorem of B. Ja. Levin on entire functions of exponential type, which is discussed in §3. The proof is completed in §4.

The result described above is clearly a special case of a general result, concerning the initial-"eigenfunction" problem for a differential expression of arbitrary order n, in which the integration contour $\partial \cdot P$ is replaced by a suitable polygon with n vertices. The reader will have no trouble formulating this more general result and supplying a proof following the model presented here. (Also, further jump relations, other than (3) are undoubtedly present and can be derived by the method of §4.)

For n = 2, the contour collapses to a slit, and the contour integration is identical to integration of the jump of the integrand (across the slit) over an interval. The representation then becomes the Volterra transformation operator which figures in the theory of the inverse scattering problem for Sturm-Liouville operators, as outlined in [2]. The jump formula analogous to (3) is well known in that context.

§2. Some Elementary Estimates.

We consider fourth-order differential expressions on the half-axis $[0,\infty)$ of the form

$$L = \frac{d^4}{dx^4} + \frac{d}{dx}\left(q\frac{d}{dx}\right) + r \tag{5}$$

with $q \in C^1([0,\infty))$, $r \in C^0([0,\infty))$. We denote by L_o the expression with $q \equiv r \equiv 0$ i.e., the fourth-derivative operator. We wish to estimate the solutions of the initial value problems

$$\begin{aligned} &L\phi_\mu = k^4\phi_\mu \ , \ \phi_\mu^{(\nu)}(0) = \delta_{\mu-1}^{\nu} \\ &\nu = 0,1,2,3 \\ &\mu = 1,2,3,4 \ , \ k \in \mathbb{C} \ . \end{aligned} \tag{6}$$

To do this we view L as a perturbation of L_0. Denote the solutions of (6) with L_0 in place of L by $\phi_\mu^0(\cdot,k)$. We have

$$\begin{aligned} \phi_1^0(x,k) &= \frac{1}{2}(\cosh kc + \cos kx) \\ \phi_2^0(x,k) &= \frac{1}{2k}(\sinh kx + \sin kx) \\ \phi_3^0(x,k) &= \frac{1}{2k^2}(\cosh kx - \cos kx) \\ \phi_4^0(x,k) &= \frac{1}{2k^3}(\sinh kx - \sin kx) \end{aligned}$$

Using variation of parameters, we obtain the following integral operation for the differences $\psi_\mu(x,k) = \phi_\mu(x,k) - \phi_\mu^0(x,k)$:

$$\begin{aligned} \psi_\mu(x,k) &+ \int_0^x G(x,y,k)\psi_\mu(y,k)dy \\ &+ \int_o^x G(x,y,k)\phi^0{}_\mu(y,k)dy = 0 \end{aligned} \tag{7}$$

where
$$\begin{aligned} G(x,y,k) = &\frac{-1}{2k}\, q(y)\,(\sinh k(x-y) + \sin k(x-y)) \\ &+ \frac{1}{2k^2}\frac{dq}{dy}(y)\,(\cosh k(x-y) - \cos k(x-y)) \\ &+ \frac{1}{2k^3}\, r(y)\,(\sinh k(x-y) - \sin k(x-y)) \end{aligned}$$

The method of successive approximations converges for the integral equation (7) since it is of Volterra type. In the process, we obtain the estimate for each $x_o \geq 0$

$$|\psi_\mu(x,k)| \leq C\,|k|^{-\mu} e^{px} \qquad 0 \leq x \leq x_o$$

where $p = \max\{|\mathrm{Re}k|,\ |\mathrm{Im}k|\}$ and the constant C generally depends on x_o. Similar estimates follow for the derivatives of ψ_μ.

§3. Some Facts from the Theory of Entire Functions.

We collect here some notions and facts from the theory of entire functions, which will be used in the next section. Proofs may be found in the monograph of B. Ja. Levin [5], Chapter I (§§ 15, 16, 19, 20). We also state a slight strengthening of Levin's generalization of the Paley-Wiener Theorem (Appendix I, §3 of [5]).

An entire function of f is of exponential type is there is $A \geq 0$ so that

$$|f(z)| < e^{A|z|}, \quad z \in \quad .$$

The (exponential) type of such a function is the number

$$\sigma = \limsup_{r \to \infty} \frac{\log M_f(r)}{r}$$

where we have used the *common notation*

$$M_f(r) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})| .$$

Otherwise put, for any $\varepsilon > 0$, there is $r > 0$ sufficiently large so that if $|z| > r$

$$|f(z)| \leq e^{(\sigma+\varepsilon)|z|}$$

but there are numbers z of arbitrarily large modulus for which

$$|f(z)| \geq e^{(\sigma-\varepsilon)|z|} .$$

It follows that if an entire function f obeys an estimate of the form

$$|f(z)| \leq C\, e^{k|z|}, \quad |z| \text{ large}$$

then f is of exponential type $\leq k$. Denote by E the class of entire functions of exponential type. The indicator function h_f of an entire function f of exponential type measures the growth of f in various directions in the complex plane. It is defined by

$$h_f(\theta) = \limsup_{r \to \infty} \frac{\log|f(re^{i\ })|}{r}$$

The indicator function enjoys the following properties:

(i) if f , $g \in E$, then

$$h_{f+g}(\theta) < \max\{h_f(\theta), h_g(\theta)\}$$

$$h_{fg}(\theta) < h_f(\theta) + h_g(\theta)$$

Recall that the support function $k(\theta)$ of a convex set G in the plane is the infinum of the signed distances from the origin to half-planes perpendicular to the ray $\arg z = \theta$, which have no points in common with the interior of G (see fig. 1).

(ii) for $f \in E$, H_f is the support function of a convex set I_f , called the <u>indicator diagram</u> of f .

(iii) The indicator diagram of $f(z) = e^{\lambda z}$ is the singleton set $\{\overline{\lambda}\}$

(iv) If $h_f(\theta) \le h_g(\theta)$, $0 \le \theta \le 2\pi$, then the indicator of f is contained in the indicator diagram of g .

(v) The indicator diagram of a sum $f_1 + \dots + f_k$ of functions $f_1 \dots . f_k \in E$ is contained in the smallest convex set containing the union $\bigcup_{i=1}^{k} I_{f_i}$.

This follows from (i) .

(vi) Suppose $f_1, \dots . f_k \in E$, and denote by L the smallest convex set containing all points which are extreme points of precisely one of the sets $I_{f_1}, \dots, I_{f_k}$. Then $K \supset I_{f_1 + \dots + f_k}$.

(vii) The indicator diagram of the exponential sum $f(z) = e^{\lambda_1 z} + e^{\lambda_2 z} + \dots + e^{\lambda_k z}$, with the λ's distinct, is the polygon with vertices $\overline{\lambda}_1, \overline{\lambda}_2, \dots . \overline{\lambda}_k$.

(viii) Suppose that the exponential type of f is > 0. The the indicator diagram of $e^{\lambda z} f(z)$ is obtained from the indicator diagram of f by translation through the vector $\overline{\lambda}$.

The next collection of facts centers around the <u>Borel Transform</u>.

If $f \in E$,

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

then the Borel transform $\hat{f}$ of f is defined by the series

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}}$$

(ix) The function of $\hat{f}(\zeta)$ is holomorphic in the domain $|\zeta| > \sigma$, where σ is the exponential type of f. In particular, $\hat{f}$ is holomorphic at infinity.

(x) The Borel transform is additive and complex-homogeneous, and the Borel transform of the function $g(z) = e^{\lambda z}$ is

$$\hat{g}(\zeta) = \frac{1}{\zeta-\lambda} .$$

(xi) Denote by γ_θ the ray $\arg z = \theta$. Suppose $f \in E$. Then the integral

$$\int_{\gamma_\theta} dz\, e^{-z\zeta} f(z)$$

converges absolutely and uniformly in the domain $\{\zeta\colon \operatorname{Re} \zeta e^{i\theta} > h_f(\theta) + \varepsilon\}$ to $\hat{f}(\zeta)$. Also, if C is any circle of radius $>\sigma$ = exponential type of f, then

$$f(z) = \frac{1}{2\pi i} \int_C d\zeta \quad e^{z\zeta}\, \hat{f}(\zeta)$$

The <u>conjugate diagram</u> of $f \in E$ is the convex hull of the singularities of $\hat{f}$. It obtains its name from the following remarkable theorem of Polyá:

(xii) The conjugate diagram is a function in E is the reflection about the real axis of its indicator diagram.

We will therefore denote the conjugate diagram of f by $\overline{I}_f$.

We come now to Levin's generalized Paley-Wiener Theorem ([5], Appendix I, §3).

<u>Theorem 3.1</u> [Levin]. Suppose $f \in E$, and suppose that the conjugate diagram I_f of f is contained in the convex polygon P. Denote by $\arg z = \theta_j$, $j = 1,\dots,n$, the rays parallel to the normals of the sides of P, and suppose that the functions

$$f_j(x) = f(xe^{-i\theta_j})e^{-k(\theta_j)x}$$

are square-integrable on the half-axis $0 < x < \infty$, where $k(\theta)$ is the support function of P. Then the Borel transform f is holomorphic outside P, and its boundary values on the perimeter ∂P define a function in $L^2(\partial P)$. Moreover,

$$f(z) = \frac{1}{2\pi i} \int_{\partial P} d\zeta \; e^{z\zeta} \hat{f}(\zeta)$$

The proof is given in Levin's book [5], Appendix I, §3. We will also need the following theorem, whose proof is similar to that of Theorem 3.1.

Theorem 3.2. Let the hypotheses be the same as in the previous theorem, except that the functions $f_j(x) = f(xe^{-i\theta_j})e^{-k(\theta_j)x}$ are assumed to be absolutely integrable. Then the Borel transform $\hat{f}$ assumes continuous boundary values on ∂P.

§4. Proof of the Representation Theorem

To derive the integral representation, we first re-interpret the estimates of §2. These estimates read

$$|\psi_1(x,k)| \le C|k|^{-1}e^{px} , \quad |\psi_3(x,k)| \le C|k|^{-3}e^{px}$$

where $p = \max\{|\mathrm{Re}\ k|, |\mathrm{Im}\ k|\}$, and

$$\psi_1(x,k) = \phi_1(x,k) - \frac{1}{2}(\cosh kx + \cos kx)$$

$$\psi_3(x,k) = \phi_3(x,k) - \frac{1}{2k^2}(\cosh kx - \cos kx) \quad .$$

Let $\phi(x,k)$ be the solution to the initial value problem

$$L\phi = k^4\phi , \quad \phi(o,h) = 1 , \quad \phi'(o,k) = 0, \ \phi''(o,k) = -k^2, \ \phi'''(o,k) = 0$$

and set

$$\psi(x,k) = \phi(x,k) - \cos kx$$

$$= \psi_1(x,k) - k^2\psi_3(x,k) .$$

We obtain the estimate

$$|\psi(x,k)| \le C|k|^{-1}e^{px} \tag{8}$$

in each finite x-interval. All of the above estimates hold for k sufficiently large, and similar estimates fold for x-derivatives up to order three.

Define

$$h(x,\theta) = x \max\{|\cos\theta|, |\sin\theta|\} , \quad 0 \le \theta \le 2\pi .$$

If we set $k = re^{i\theta}$, we can rewrite (8) in the form

$$|\psi(x,k) \le C\, r^{-1}\, e^{rh(x,\theta)} , \quad \text{large} . \tag{9}$$

We can therefore conclude that $\psi(x,k)$ is an entire function of exponential type $\le x$, and moreover that the indicator function of $\psi(x,\cdot)$ is bounded above by $h(x,\theta)$. According to §3, property (iv), the indicator diagram of $\psi(x,\cdot)$ is contained in the diamond $P(x)$ with vertices θx, ix, which is the convex figure of which $h(x,\theta)$ is the support function. Since $\overline{P}(x) = P(x)$, the same is true of the conjugate diagram of $\psi(x,\cdot)$. Moreover, estimate (9) shows that the function

$$e^{-rh(x,\theta)}\psi(x,re^{i\theta})$$

is square-integrable as a function of r , $0 \le r \le \infty$ for $0 \le \quad \le 2\pi$. It follows from Theorem 3.1 that the Borel transform $A(x,\cdot)$ of $\psi(x,\cdot)$ is homomorphic in the experior of $P(x)$, assumes square-integrable boundary values on $\partial P(x)$, and

$$\psi(x,k) = \frac{1}{2\pi i} \int_{\partial P(x)} dz\, e^{kz} A(x,z)$$

so that the function $\phi(x,k)$ admits the integral representation

$$\phi(x,k) = \cos kx + \frac{1}{2\pi i} \int_{\partial P(x)} dz\, e^{kz} A(x,z) .$$

We will now show that the kernel $A(x,z)$ is a continuous function of $z \in \partial P(x)$, except at the vertices $\pm ix$. To do this, derive from the integral equation (7) for $\psi_\mu, \mu = 1,2,3,4$, an integral equation for ψ :

$$\psi(x,k) + \int_0^x dy\, G(x,y,k)\psi(y,k) + \int_0^x dy\, G(x,y,k)\cos ky = 0$$

where $G(x,y,k)$ *is defined in* §2, *and satisfied the estimate*

$$|G(x,y,k)| \le M(y)\, r^{-1} e^{rh(x-y,\theta)}$$

$$M(y) = \max\{|q(y)|, |q'(y)|, |r(y)|\}$$

and $k = re^{i\theta}$.

Since $e^{rh(x-y,\theta)}\,|\cos ky| \le e^{rh(x,\theta)}$ and $h(x-y,\theta) + h(y,\theta) = h(x,\theta)$, we obtain for successive terms in the Neumann series

$$U_1(x,k) = -\int_0^x dy\, G(x,y,k)\cos ky$$

$$U_{n+1}(x,k) = -\int_0^x dy\, G(x,y,k)U_n(y,k)$$

the estimate

$$|U_n(x,k)| < \left(\frac{M}{r}\right)^n e^{rh(x,\theta)}$$

$$M = \int_0^x dy M(y) .$$

Write

$$J_1(x,k) = \frac{-1}{2k}\int_0^x dy\, q(y)(\sinh k(x-y) + \sin k(x-y))\cos ky \ .$$

Then the above estimates imply that

$$\psi(x,k) = J_1(x,k) + J_2(x,k)$$

where $J_1(x,\cdot)$, $J_2(x,\cdot)$ are entire functions of exponential type with indicator diagrams contained in $P(x)$, and

$$J_2(x,k) = O(|k|^{-2} e^{rh(x,\theta)})$$

$$\text{as } k = re^{i\theta} \to \infty \ .$$

It follows from Theorem 3.2 that

$$A(x,z) = \hat{J}_1(x,z) + \hat{J}_2(x,z)$$

where $\hat{J}_2(x,z)$ is continuous in $z \in \partial P(x)$. We therefore need only show that $\hat{J}_1(x,z)$ is continuous except at $z = \pm ix$. To do this write

$$\frac{1}{k}(\sinh kx + \sin kx) = \int_0^x dt(\cosh kt + \cos kt)$$

so that

$$J_1(x,k) = -\frac{1}{2}\int_0^x dy \int_0^{x-y} dt\, q(y)(\cosh kt + \cos kt)\cos ky$$

$$= -\frac{1}{4}\int_0^x dt \int_0^{x-t} dy\, q(y)\, [e^{(t+iy)k} + e^{(t-iy)k} + e^{(-t+iy)k}$$

$$+ e^{(-t-iy)k} + e^{i(t+y)k} + e^{i(t-y)k} + e^{i(-t+y)k}$$

$$+ e^{i(-t-y)k}]$$

Since the indicator of $J_1(x,\cdot)$ is bounded by $h(x, 0)$, the integral

$$\int_0^\infty dk\, e^{-kz} J_1(x,k)$$

converges for $\operatorname{Re} z$ sufficiently large to $\hat{J}_1(x,\cdot)$, the Borel transform of $J_1(x,\cdot)$. It is easy to justify changing the order of integration. Using property (x) of §3 we obtain

$$\hat{J}_1(x,z) = -\frac{1}{4}\int_0^x dt \int_0^{x-t} dy\, q(y)\, \Big[\frac{1}{z-(t+iy)} + \frac{1}{z-(t-iy)}$$

$$+ \frac{1}{z-(-t+iy)} + \frac{1}{z+(t+iy)} + \frac{1}{z-i(t+y)} + \frac{1}{z-i(t-y)}$$

$$+ \frac{1}{z-i(-t+y)} + \frac{1}{z+i(t+y)}\Big],$$

valid for $\mathrm{Re}\, z$ sufficiently large. An elementary normal families argument allows us to continue the right hand side in the exterior of P, so that the above representation is valid in $\mathbb{C}\backslash\overline{P}$. We write the right hand side as a sum of eight integrals. The sum of the first four defines a function holomorphic in $\mathbb{C}\backslash\overline{P}$ and continuous on $(\mathbb{C}\backslash P)\cup\partial P$, in view of the following.

Lemma. Suppose that $Q(z)$ is bounded, absolutely integrable, and has compact support contained in the closed upper half-plane. Then the integral

$$\iint_{\mathrm{Im}\, z\geq 0} dz d\bar{z}\, \frac{Q(z)}{z-\zeta}$$

converges absolutely and defines a continuous function of ζ in the closed lower half-plane $\{\zeta\colon \mathrm{Im}\,\zeta \leq 0\}$, holomorphic for $\mathrm{Im}\,\zeta < 0$.

The sixth and seventh integrals can be shown to define continuous functions up to the boundary ∂P: they are in fact holomorphic in the complement of the segment $[-ix, ix]$ on the imaginary axis, and have integrable singularities at the vertices $\pm ix$. The fifth and eighth integrals are also holomorphic in the complement of $[-ix, ix]$, but have non-integrable singularities at $\pm ix$. We have thus proven the assertion regarding the continuity of $A(x,z)$ in z.

We can also compute the jumps of $A(x,z)$ at the vertices $\pm ix$, which are precisely the jumps of the fifth and eighth integrals above. The computation is standard; see for instance [1], pp. 73-75. We obtain

$$\lim_{\delta\to 0} (A(x, \pm ix-\delta) - A(x, \pm ix+\delta)) = -\frac{\pi}{4}\int_0^x dt\, q(t)$$

Since $\phi(x,k)$ is an even function of k, $A(x,z)$ is an odd function of z : this follows from the inversion formula for the Borel transform (§6, property (ix)). Also, the reality of $\phi(x,k)$ for k real implies that $A(x,\bar{z}) = -\overline{A(x,z)}$ for $z \in \partial P(x)$.

This concludes the proof of the representation theorem.

References

1. H. Bremermann, Distributions, Complex Variables, and Fourier Transforms, Reading, Mass, 1965.

2. L.D. Fadeev, "The Inverse Problem in the Quantum Theory of Scattering", Journal of Mathematicsl Physics 4(1963), p. 72.

3. B. Ja.Levin, Distribution of Zeroes of Entire Functions, Providence R.I., 1964.

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20. ABSTRACT - Cont'd.

$$L\phi = k^4\phi, \quad L = \frac{d^4}{dx^4} + \frac{d}{dx}\left(q\,\frac{d}{dx}\right) + r$$

$$\phi(o) = 1,\ \phi'(o) = 0,\ \phi''(o) = -k^2,\ \phi'''(o) = 0\ .$$

The solution ~~for complex k~~ are ~~is~~ expressed as an inverse-Laplace-Borel transform. Jump formulae are obtained relating the representing kernel directly to the coefficients of L. The result admits obvious generalization to operators of arbitrary order.

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